

NEW ESTIMATES ON GENERALIZATION OF SOME INTEGRAL INEQUALITIES FOR (α, m) -CONVEX FUNCTIONS

İMDAT İŞCAN

ABSTRACT. In this paper, we derive new estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae for functions whose derivatives in absolute value at certain power are (α, m) -convex.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality is well known in the literature as Hermite-Hadamard integral inequality

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The class of (α, m) -convex functions was first introduced In [2], and it is defined as follows:

The function $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$, is said to be (α, m) -convex where $(\alpha, m) \in [0, 1]^2$, if we have

$$(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

It can be easily that for $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ one obtains the following classes of functions: increasing, α -starshaped, starshaped, m -convex, convex, α -convex.

Denote by $K_m^\alpha(b)$ the set of all (α, m) -convex functions on $[0, b]$ for which $f(0) \leq 0$. For recent results and generalizations concerning (α, m) -convex functions (see [1, 2, 3, 4, 5, 10]).

The following inequality is well known in the literature as Simpson's inequality .

Let $f : [a, b] \rightarrow \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:

$$\left| \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b-a)^2.$$

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In recent years many authors have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [6, 7, 8, 9].

In this paper, in order to provide a unified approach to establish midpoint inequality, trapezoid inequality and Simpson's inequality for functions whose derivatives in absolute value at certain power are (α, m) -convex, we derive a general integral identity for convex functions.

2. MAIN RESULTS

In order to generalize the classical Trapezoid, midpoint and Simpson type inequalities and prove them, we need the following Lemma:

Lemma 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[ma, mb]$, where $m \in (0, 1]$, $ma, mb \in I$ with $a < b$, then for $\theta, \lambda \in [0, 1]$ the following equality holds:*

$$\begin{aligned} & (1 - \theta) (\lambda f(ma) + (1 - \lambda) f(mb)) + \theta f((1 - \lambda) ma + \lambda mb) - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x) dx \\ &= m(b-a) \left[-\lambda^2 \int_0^1 (t - \theta) f'(tma + (1-t)[(1-\lambda)ma + \lambda mb]) dt \right. \\ & \quad \left. + (1-\lambda)^2 \int_0^1 (t - \theta) f'(tmb + (1-t)[(1-\lambda)ma + \lambda mb]) dt \right]. \end{aligned}$$

A simple proof of the equality can be done by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader.

Theorem 1. *Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[ma, mb]$, where $m \in (0, 1]$, $ma, mb \in I^\circ$ with $a < b$ and $\theta, \lambda \in [0, 1]$. If $|f'|^q$ is (α, m) -convex on $[ma, mb]$, for $\alpha \in [0, 1]$, $q \geq 1$ then the following inequality holds:*

$$\begin{aligned} & \left| (1 - \theta) (\lambda f(ma) + (1 - \lambda) f(mb)) + \theta f(mC) - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x) dx \right| \\ (2.1) \leq & m(b-a) A_1^{1-\frac{1}{q}}(\theta) \min \{B_1(\theta, \lambda, \alpha, q, m), B_2(\theta, \lambda, \alpha, q, m)\}, \end{aligned}$$

where

$$\begin{aligned} B_1(\theta, \lambda, \alpha, q, m) &= \left\{ \lambda^2 (|f'(ma)|^q A_2(\theta, \alpha) + m |f'(C)|^q A_3(\theta, \alpha))^{\frac{1}{q}} \right. \\ & \quad \left. + (1-\lambda)^2 (|f'(mb)|^q A_2(\theta, \alpha) + m |f'(C)|^q A_3(\theta, \alpha))^{\frac{1}{q}} \right\}, \\ B_2(\theta, \lambda, \alpha, q, m) &= \left\{ \lambda^2 (|f'(mC)|^q A_4(\theta, \alpha) + m |f'(a)|^q A_5(\theta, \alpha))^{\frac{1}{q}} \right. \\ & \quad \left. + (1-\lambda)^2 (|f'(mC)|^q A_4(\theta, \alpha) + m |f'(b)|^q A_5(\theta, \alpha))^{\frac{1}{q}} \right\}, \end{aligned}$$

$$\begin{aligned}
 A_1(\theta) &= \theta^2 - \theta + \frac{1}{2}, \\
 A_2(\theta, \alpha) &= \frac{2\theta^{\alpha+2}}{(\alpha+1)(\alpha+2)} - \frac{\theta}{\alpha+1} + \frac{1}{\alpha+2}, \\
 A_3(\theta, \alpha) &= \theta^2 - \frac{2\theta^{\alpha+2}}{(\alpha+1)(\alpha+2)} - \frac{\alpha\theta}{\alpha+1} + \frac{\alpha}{2(\alpha+2)}, \\
 A_4(\theta, \alpha) &= \frac{2(1-\theta)^{\alpha+2}}{(\alpha+1)(\alpha+2)} - \frac{1-\theta}{\alpha+1} + \frac{1}{\alpha+2}, \\
 A_5(\theta, \alpha) &= (1-\theta)^2 - \frac{2(1-\theta)^{\alpha+2}}{(\alpha+1)(\alpha+2)} - \frac{\alpha(1-\theta)}{\alpha+1} + \frac{\alpha}{2(\alpha+2)},
 \end{aligned}$$

and $C = (1-\lambda)a + \lambda b$.

Proof. Suppose that $C = (1-\lambda)a + \lambda b$. From Lemma 1 and using the properties of modulus and the well known power mean inequality, we have

$$\begin{aligned}
 & (1-\theta)(\lambda f(ma) + (1-\lambda)f(mb)) + \theta f((1-\lambda)ma + \lambda mb) - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x) dx \\
 & \leq m(b-a) \left[\lambda^2 \int_0^1 |t-\theta| |f'(tma + (1-t)mC)| dt \right. \\
 & \quad \left. + (1-\lambda)^2 \int_0^1 |t-\theta| |f'(tmb + (1-t)mC)| dt \right] \\
 & \leq m(b-a) \left\{ \lambda^2 \left(\int_0^1 |t-\theta| dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |t-\theta| |f'(tma + (1-t)mC)|^q dt \right)^{\frac{1}{q}} \right. \\
 (2.2) \quad & \left. + (1-\lambda)^2 \left(\int_0^1 |t-\theta| dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |t-\theta| |f'(tmb + (1-t)mC)|^q dt \right)^{\frac{1}{q}} \right\}
 \end{aligned}$$

Since $|f'|^q$ is (α, m) -convex on $[a, b]$, we know that for $t \in [0, 1]$

$$|f'(tma + (1-t)mC)|^q \leq t^\alpha |f'(ma)|^q + m(1-t^\alpha) |f'(C)|^q,$$

and

$$|f'(tmb + (1-t)mC)|^q \leq t^\alpha |f'(mb)|^q + m(1-t^\alpha) |f'(C)|^q.$$

Hence, by simple computation

$$\begin{aligned}
 (2.3) \quad & \int_0^1 |t-\theta| t^\alpha |f'(ma)|^q + m(1-t^\alpha) |f'(C)|^q dt \\
 & = |f'(ma)|^q A_2(\theta, \alpha) + m |f'(C)|^q A_3(\theta, \alpha)
 \end{aligned}$$

$$\begin{aligned}
(2.4) \quad & \int_0^1 |t - \theta| t^\alpha |f'(mb)|^q + m(1 - t^\alpha) |f'(C)|^q dt \\
& = |f'(mb)|^q A_2(\theta, \alpha) + m |f'(C)|^q A_3(\theta, \alpha)
\end{aligned}$$

and

$$(2.5) \quad \int_0^1 |t - \theta| dt = \theta^2 - \theta + \frac{1}{2}.$$

Thus, using (2.3)-(2.5) in (2.2), we obtain the following inequality

$$\begin{aligned}
(2.6) \quad & \left| (1 - \theta) (\lambda f(ma) + (1 - \lambda) f(mb)) + \theta f(mC) - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x) dx \right| \\
& \leq m(b-a) A_1^{1-\frac{1}{q}}(\theta) \left\{ (|f'(ma)|^q A_2(\theta, \alpha) + m |f'(C)|^q A_3(\theta, \alpha))^{\frac{1}{q}} \right. \\
& \quad \left. + (|f'(mb)|^q A_2(\theta, \alpha) + m |f'(C)|^q A_3(\theta, \alpha))^{\frac{1}{q}} \right\}.
\end{aligned}$$

In the inequality (2.2), if we use equalities

$$\int_0^1 |t - \theta| |f'(tma + (1-t)mC)|^q dt = \int_0^1 |1 - \theta - t| |f'(tmC + (1-t)ma)|^q dt$$

and

$$\int_0^1 |t - \theta| |f'(tmb + (1-t)mC)|^q dt = \int_0^1 |1 - \theta - t| |f'(tmC + (1-t)mb)|^q dt,$$

by similar process, since $|f'|^q$ is (α, m) -convex on $[a, b]$, for $t \in [0, 1]$

$$|f'(tmC + (1-t)ma)|^q \leq t^\alpha |f'(mC)|^q + m(1 - t^\alpha) |f'(a)|^q$$

and

$$|f'(tmC + (1-t)mb)|^q \leq t^\alpha |f'(mC)|^q + m(1 - t^\alpha) |f'(b)|^q.$$

Similarly, by simple computation

$$\begin{aligned}
(2.7) \quad & \int_0^1 |1 - \theta - t| t^\alpha |f'(mC)|^q + m(1 - t^\alpha) |f'(a)|^q dt \\
& = |f'(mC)|^q A_4(\theta, \alpha) + m |f'(a)|^q A_5(\theta, \alpha)
\end{aligned}$$

$$\begin{aligned}
(2.8) \quad & \int_0^1 |1 - \theta - t| t^\alpha |f'(mC)|^q + m(1 - t^\alpha) |f'(b)|^q dt \\
& = |f'(mC)|^q A_4(\theta, \alpha) + m |f'(b)|^q A_5(\theta, \alpha)
\end{aligned}$$

Thus, using (2.3),(2.7) and (2.8) in (2.2), we have the following inequality

$$\begin{aligned}
 & \left| (1-\theta)(\lambda f(ma) + (1-\lambda)f(mb)) + \theta f(mC) - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x) dx \right| \\
 & \leq m(b-a) A_1^{1-\frac{1}{q}}(\theta) \left\{ (|f'(mC)|^q A_4(\theta, \alpha) + m |f'(a)|^q A_5(\theta, \alpha))^{\frac{1}{q}} \right. \\
 (2.9) \quad & \left. + (|f'(mC)|^q A_4(\theta, \alpha) + m |f'(b)|^q A_5(\theta, \alpha))^{\frac{1}{q}} \right\}.
 \end{aligned}$$

From the inequalities (2.6) and (2.9) the inequality (2.1) is obtained. This completes the proof. \square

Corollary 1. *Under the assumptions of Theorem 1 with $q = 1$*

$$\begin{aligned}
 & \left| (1-\theta)(\lambda f(ma) + (1-\lambda)f(mb)) + \theta f(mC) - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x) dx \right| \\
 & \leq m(b-a) \min \{ B_1(\theta, \lambda, \alpha, 1, m), B_2(\theta, \lambda, \alpha, 1, m) \}.
 \end{aligned}$$

Corollary 2. *Under the assumptions of Theorem 1 with $\lambda = \frac{1}{2}$ and $\theta = \frac{2}{3}$, we have*

$$\begin{aligned}
 & \left| \frac{1}{6} \left[f(ma) + 4f\left(\frac{ma+mb}{2}\right) + f(mb) \right] - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x) dx \right| \\
 & \leq m(b-a) \left(\frac{5}{18}\right)^{1-\frac{1}{q}} \min \left\{ B_1\left(\frac{2}{3}, \frac{1}{2}, \alpha, q, m\right), B_2\left(\frac{2}{3}, \frac{1}{2}, \alpha, q, m\right) \right\}.
 \end{aligned}$$

Remark 1. *In Corollary 2, if we take $\alpha = m = 1$, we obtain the following inequality*

$$\begin{aligned}
 & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq (b-a) A_1^{1-\frac{1}{q}}(\theta) \min \left\{ B_1\left(\frac{2}{3}, \frac{1}{2}, 1, q, 1\right), B_2\left(\frac{2}{3}, \frac{1}{2}, 1, q, 1\right) \right\} \\
 & \leq (b-a) A_1^{1-\frac{1}{q}}(\theta) B_2\left(\frac{2}{3}, \frac{1}{2}, 1, q, 1\right) \\
 & \leq (b-a) \left(\frac{5}{72}\right)^{1-\frac{1}{q}} \left\{ \left(\frac{29}{648} \left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{2}{81} |f'(a)|^q\right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left(\frac{29}{648} \left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{2}{81} |f'(b)|^q\right)^{\frac{1}{q}} \right\},
 \end{aligned}$$

which is the better than the inequality in [8, Theorem 10] for $s = 1$.

Corollary 3. *Under the assumptions of Theorem 1 with $\lambda = \frac{1}{2}$ and $\theta = 0$, we have*

$$\begin{aligned}
 & \left| \frac{f(ma) + f(mb)}{2} - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x) dx \right| \\
 & \leq m(b-a) \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \min \left\{ B_1\left(0, \frac{1}{2}, \alpha, q, m\right), B_2\left(0, \frac{1}{2}, \alpha, q, m\right) \right\},
 \end{aligned}$$

Corollary 4. Under the assumptions of Theorem 1 with $\lambda = \frac{1}{2}$ and $\theta = 1$, we have

$$\begin{aligned} & \left| f\left(\frac{ma+mb}{2}\right) - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x)dx \right| \\ & \leq m(b-a) \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \min \left\{ B_1\left(1, \frac{1}{2}, \alpha, q, m\right), B_2\left(1, \frac{1}{2}, \alpha, q, m\right) \right\} \end{aligned}$$

Theorem 2. Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[ma, mb]$, where $m \in (0, 1]$, $ma, mb \in I^\circ$ with $a < b$ and $\theta, \lambda \in [0, 1]$. If $|f'|^q$ is (α, m) -convex on $[ma, mb]$, for $\alpha \in [0, 1]$, $q > 1$ then the following inequality holds:

$$\begin{aligned} & \left| (1-\theta)(\lambda f(ma) + (1-\lambda)f(mb)) + \theta f(mC) - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x)dx \right| \\ & \leq m(b-a) \left(\frac{\theta^{p+1} + (1-\theta)^{p+1}}{p+1} \right)^{\frac{1}{p}} \min \{ B_3(\lambda, \alpha, q), B_4(\lambda, \alpha, q) \}, \end{aligned}$$

where

$$B_3(\lambda, \alpha, q) = \left\{ \lambda^2 E_1^{\frac{1}{q}}(\lambda, \alpha) + (1-\lambda)^2 E_2^{\frac{1}{q}}(\lambda, \alpha) \right\},$$

$$B_4(\lambda, \alpha, q) = \left\{ \lambda^2 E_3^{\frac{1}{q}}(\lambda, \alpha) + (1-\lambda)^2 E_4^{\frac{1}{q}}(\lambda, \alpha) \right\},$$

$$E_1(\lambda, \alpha) = \frac{|f'(ma)|^q + \alpha m |f'(C)|^q}{\alpha + 1},$$

$$E_2(\lambda, \alpha) = \frac{|f'(mb)|^q + \alpha m |f'(C)|^q}{\alpha + 1},$$

$$E_3(\lambda, \alpha) = \frac{|f'(mC)|^q + \alpha m |f'(a)|^q}{\alpha + 1},$$

$$E_4(\lambda, \alpha) = \frac{|f'(mC)|^q + \alpha m |f'(b)|^q}{\alpha + 1},$$

$$C = (1-\lambda)a + \lambda b \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Suppose that $C = (1-\lambda)a + \lambda b$. From Lemma 1 and by Hölder's integral inequality, we have

$$\begin{aligned} & \left| (1-\theta)(\lambda f(ma) + (1-\lambda)f(mb)) + \theta f(mC) - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x)dx \right| \leq m(b-a) \\ & \left[\lambda^2 \int_0^1 |t-\theta| |f'(tma + m(1-t)C)| dt + (1-\lambda)^2 \int_0^1 |t-\theta| |f'(tmb + m(1-t)C)| dt \right] \\ & \leq m(b-a) \left\{ \lambda^2 \left(\int_0^1 |t-\theta|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(tma + m(1-t)C)|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + (1-\lambda)^2 \left(\int_0^1 |t-\theta|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(tmb + m(1-t)C)|^q dt \right)^{\frac{1}{q}} \right\} \end{aligned}$$

$$(2.10) \quad + (1 - \lambda)^2 \left(\int_0^1 |t - \theta|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(tmb + m(1-t)C)|^q dt \right)^{\frac{1}{q}} \Bigg\}.$$

Since $|f'|^q$ is (α, m) -convex on $[a, b]$, we know that for $t \in [0, 1]$

$$|f'(tma + m(1-t)C)|^q \leq t^\alpha |f'(ma)|^q + m(1-t^\alpha) |f'(C)|^q,$$

and

$$|f'(tmb + m(1-t)C)|^q \leq t^\alpha |f'(mb)|^q + m(1-t^\alpha) |f'(C)|^q.$$

Hence, by simple computation

$$(2.11) \quad \int_0^1 t^\alpha |f'(ma)|^q + m(1-t^\alpha) |f'(C)|^q dt = \frac{|f'(ma)|^q + \alpha m |f'(C)|^q}{\alpha + 1},$$

$$(2.12) \quad \int_0^1 t^\alpha |f'(mb)|^q + m(1-t^\alpha) |f'(C)|^q dt = \frac{|f'(mb)|^q + \alpha m |f'(C)|^q}{\alpha + 1},$$

and

$$(2.13) \quad \int_0^1 |t - \theta|^p dt = \frac{\theta^{p+1} + (1-\theta)^{p+1}}{p+1}$$

thus, using (2.11)-(2.13) in (2.10), we obtain the following inequality

$$(2.14) \quad \begin{aligned} & \left| (1-\theta)(\lambda f(ma) + (1-\lambda)f(mb)) + \theta f(mC) - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x) dx \right| \\ & \leq m(b-a) \left(\frac{\theta^{p+1} + (1-\theta)^{p+1}}{p+1} \right)^{\frac{1}{p}} \left\{ \lambda^2 \left(\frac{|f'(ma)|^q + \alpha m |f'(C)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right. \\ & \quad \left. (1-\lambda)^2 \left(\frac{|f'(mb)|^q + \alpha m |f'(C)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Similarly

$$(2.15) \quad \begin{aligned} \int_0^1 |f'(tma + (1-t)mC)|^q dt &= \int_0^1 |f'(tmC + (1-t)ma)|^q dt \\ &\leq \int_0^1 t^\alpha |f'(mC)|^q + m(1-t^\alpha) |f'(a)|^q dt \\ &= \frac{|f'(mC)|^q + \alpha m |f'(a)|^q}{\alpha + 1} \end{aligned}$$

and

$$\begin{aligned}
 \int_0^1 |f'(tmb + (1-t)mC)|^q dt &= \int_0^1 |f'(tmC + (1-t)mb)|^q dt \\
 &\leq \int_0^1 t^\alpha |f'(mC)|^q + m(1-t^\alpha) |f'(b)|^q dt \\
 (2.16) \qquad &= \frac{|f'(mC)|^q + \alpha m |f'(b)|^q}{\alpha + 1}.
 \end{aligned}$$

By using (2.11), (2.15) and (2.16) in (2.10), we get the following inequality

$$\begin{aligned}
 &\left| (1-\theta)(\lambda f(ma) + (1-\lambda)f(mb)) + \theta f(mC) - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x) dx \right| \\
 &\leq m(b-a) \left(\frac{\theta^{p+1} + (1-\theta)^{p+1}}{p+1} \right)^{\frac{1}{p}} \left\{ \lambda^2 \left(\frac{|f'(mC)|^q + \alpha m |f'(a)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right. \\
 (2.17) \quad &\left. (1-\lambda)^2 \left(\frac{|f'(mC)|^q + \alpha m |f'(b)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right\}.
 \end{aligned}$$

From the inequalities (2.14) and (2.17) the inequality (??) is obtained. This completes the proof. \square

Corollary 5. *Under the assumptions of Theorem 2 with $\lambda = \frac{1}{2}$ and $\theta = \frac{2}{3}$, we have*

$$\begin{aligned}
 &\left| \frac{1}{6} \left[f(ma) + 4f\left(\frac{ma+mb}{2}\right) + f(mb) \right] - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x) dx \right| \\
 &\leq \frac{m(b-a)}{12} \left(\frac{2^{p+1} + 1}{3(p+1)} \right)^{\frac{1}{p}} \min \left\{ E_1^{\frac{1}{q}}\left(\frac{1}{2}, \alpha\right) + E_2^{\frac{1}{q}}\left(\frac{1}{2}, \alpha\right), E_3^{\frac{1}{q}}\left(\frac{1}{2}, \alpha\right) + E_4^{\frac{1}{q}}\left(\frac{1}{2}, \alpha\right) \right\},
 \end{aligned}$$

where

$$\begin{aligned}
 E_1\left(\frac{1}{2}, \alpha\right) &= \frac{|f'(ma)|^q + \alpha m |f'\left(\frac{a+b}{2}\right)|^q}{\alpha + 1}, \\
 E_2\left(\frac{1}{2}, \alpha\right) &= \frac{|f'(mb)|^q + \alpha m |f'\left(\frac{a+b}{2}\right)|^q}{\alpha + 1}, \\
 E_3\left(\frac{1}{2}, \alpha\right) &= \frac{\left|f'\left(\frac{m(a+b)}{2}\right)\right|^q + \alpha m |f'(a)|^q}{\alpha + 1}, \\
 E_4\left(\frac{1}{2}, \alpha\right) &= \frac{\left|f'\left(\frac{m(a+b)}{2}\right)\right|^q + \alpha m |f'(b)|^q}{\alpha + 1},
 \end{aligned}$$

Remark 2. In Corollary 5, if we take $\alpha = m = 1$, then we obtain the following inequality

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \left(\frac{b-a}{12} \right) \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \\ \times 2 \cdot \min \left\{ \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}}, \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right\},$$

which is the better than the inequality in [8, Corollary 3]

Corollary 6. Under the assumptions of Theorem 2 with $\lambda = \frac{1}{2}$ and $\theta = 0$, we have

$$\left| \frac{f(ma) + f(mb)}{2} - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x)dx \right| \\ \leq \frac{m(b-a)}{4} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \min \left\{ E_1^{\frac{1}{q}}\left(\frac{1}{2}, \alpha\right) + E_2^{\frac{1}{q}}\left(\frac{1}{2}, \alpha\right), E_3^{\frac{1}{q}}\left(\frac{1}{2}, \alpha\right) + E_4^{\frac{1}{q}}\left(\frac{1}{2}, \alpha\right) \right\}.$$

Corollary 7. Under the assumptions of Theorem 2 with $\lambda = \frac{1}{2}$ and $\theta = 1$, we have

$$\left| f\left(\frac{m(a+b)}{2}\right) - \frac{1}{m(b-a)} \int_{ma}^{mb} f(x)dx \right| \\ \leq \frac{m(b-a)}{4} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \min \left\{ E_1^{\frac{1}{q}}\left(\frac{1}{2}, \alpha\right) + E_2^{\frac{1}{q}}\left(\frac{1}{2}, \alpha\right), E_3^{\frac{1}{q}}\left(\frac{1}{2}, \alpha\right) + E_4^{\frac{1}{q}}\left(\frac{1}{2}, \alpha\right) \right\}.$$

REFERENCES

- [1] Bakula, M.K., Ozdemir, M.E. and Pecaric, J.: *Hadamard type inequalities for m -convex and (α, m) -convex functions*, J. Inequal. Pure Appl. Math. 9, no.4, Article 96, p. 12, 2008. [Online: <http://jipam.vu.edu.au>].
- [2] Miheşan, V.G.: *A generalization of the convexity*, Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, Romania (1993).
- [3] Ozdemir, M.E., Avci, M. and Kavurmaci, H.: *Hermite-Hadamard-type inequalities via (α, m) -convexity*, Computers and Mathematics with Applications 61, 2614-2620, 2011.
- [4] Ozdemir, M.E., Kavurmaci, H. and Set, E.: *Ostrowski's type inequalities for (α, m) -convex functions*, Kyungpook Math. J. 50, 371-378, 2010.
- [5] Ozdemir, M.E., Set, E. and Sarikaya, M.Z.: *Some new Hadamard's type inequalities for co-ordinated m -convex and (α, m) -convex functions*, Hacettepe Journal of Mathematics and Statistics volume 40 (2), 219-229, 2011.
- [6] J. Park, *Hermite-Hadamard and Simpson-like type inequalities for differentiable (α, m) -convex mappings*, Int. Journal of Math. and Mathematical Sciences, vol. 2012, Article ID 809689, 12 pages.
- [7] Sarikaya, M.Z., Aktan, N.: *On the generalization of some integral inequalities and their applications*, Mathematical and Computer Modelling, 54 (2011) 2175-2182.
- [8] Sarikaya, M.Z., Set, E., Özdemir, M.E.: *On new inequalities of Simpson's type for s -convex functions*, Computers and Mathematics with Applications 60 (2010) 2191-2199.
- [9] Sarikaya, M.Z., Set, E., Özdemir, M.E.: *On new inequalities of Simpson's type for convex functions*, RGMIA Res. Rep. Coll. 13 (2) (2010) Article 2.
- [10] Set, E., Sardari, M., Ozdemir, M.E. and Roojin, J.: *On generalizations of the Hadamard inequality for (α, m) -convex functions*, RGMIA Res. Rep. Coll. 12(4), Article 4, 2009.

DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES,, GİRESUN UNIVERSITY,
28100, GİRESUN, TURKEY.

E-mail address: `imdat.iscan@giresun.edu.tr`